

Numerical Analysis — FMN011 — 110603

The exam lasts 4 hours. A minimum of 35 points out of the total 70 are required to get a passing grade. These points will be added to those obtained in your two home assignments, and the final grade is based on your total score.

Justify all your answers and write down all important steps. Unsupported answers will be disregarded.

During the exam you are allowed a pocket calculator, but no textbook, lecture notes or any other electronic or written material.

- (4p)** Suppose matrix A has condition number (relative to the ∞ norm) equal to 10^{11} . The solution to the system $Ax = b$ is $x = [1, 1, \dots, 1]^T$, but when solved in Matlab I get a relative error of magnitude 10^{-6} in the ∞ -norm. Is it reasonable to attribute this error to ill-conditioning? Justify your answer.
- (4p)** Consider the function $f(x) = x + \ln x$.
 - Plot the functions $y = x$ and $y = -\ln x$ to show f has a unique root P in $(0, \infty)$.
 - Can the root P be found using a fixed point iteration $x = g(x)$ with $g(x) = -\ln x$? Justify.
- (5p)** If $\|\cdot\|$ is a matrix norm, then $\|A\| = 0 \Rightarrow A = 0$. Find a 2×2 nonzero matrix for which the spectral radius is zero, i.e., $\rho(A) = 0$. Is the spectral radius a matrix norm? Justify. (The spectral radius is $\rho(A) = \{\max_{\lambda} |\lambda|, Ax = \lambda x, x \neq 0\}$.)
- (5p)** Consider the matrix

$$A = \begin{bmatrix} 0 & -8 & 8 & 3 \\ 0 & 4 & -4 & 3 \\ 3 & 4 & 1 & 4 \\ -4 & -3 & 8 & 3 \end{bmatrix}$$

What will the initial pivot in Gaussian elimination be if

- No pivoting is used?
 - Partial pivoting is used?
 - Scaled partial pivoting is used?
- (5p)** Prove that Newton's method $x_{n+1} = x_n - \partial F(x_n)^{-1} F(x_n)$ applied to the linear system $Ax = b$ converges to the exact solution in one single step. ($\partial F(x)$ is the Jacobian matrix, i.e., the matrix whose rows are the gradients of the rows of $F(x)$).
 - (5p)** Select the most appropriate answer.
 - If a function is interpolated at n points $\{x_1, x_2, \dots, x_n\}$, the error of the interpolation at a point $x \neq x_j$ contained in the interval of interpolation depends on

- i. the basis chosen for the interpolation
 - ii. the value of the function at x
 - iii. the number of data points
- (b) All previous computations can still be used when new data points are added in the following type of polynomial representations:
- i. Lagrange's
 - ii. Bernstein's
 - iii. Newton's
- (c) With quadratic splines, which of the following are continuous at the knots?
- i. The function.
 - ii. The function and its derivative.
 - iii. The function and its first and second derivatives.
- (d) What type of matrix is involved in the construction of a cubic spline?
- i. a symmetric matrix
 - ii. a diagonal matrix
 - iii. a diagonally dominant matrix
- (e) The Jacobi or Gauss-Seidel methods applied to $Ax = b$ will converge if
- i. A is strictly diagonally dominant
 - ii. the largest eigenvalue of A has absolute value equal to 0.5
 - iii. the initial guess is close enough to the exact solution

7. Consider the following polynomial bases:

- Monomial basis
- Lagrange basis
- Newton basis

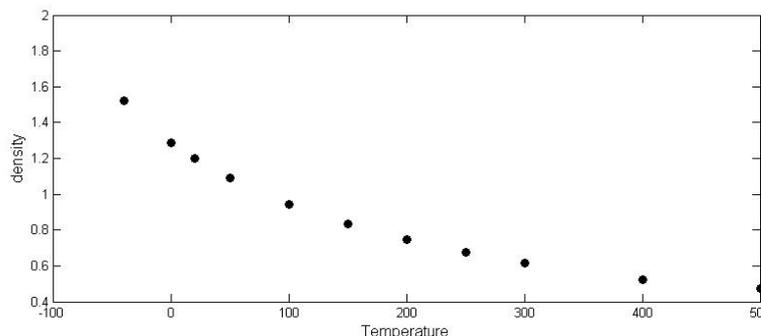
- (a) **(2p)** Describe the pattern of nonzero entries in the basis matrix used to determine the coefficients for polynomial interpolation using each of the three bases.
- (b) **(2p)** Rank the three bases according to the cost of determining the coefficients of the interpolating polynomial (from largest to smallest).
- (c) **(2p)** Which of the three methods has the best-conditioned basis matrix?

8. **(6p)** $A = QD$ with

$$Q = \begin{pmatrix} -0.39074 & -0.55664 & 0.29336 & -0.67187 \\ -0.44012 & -0.45065 & 0.24592 & 0.7367 \\ -0.51035 & -0.04114 & -0.85787 & -0.043687 \\ -0.62703 & 0.69669 & 0.34281 & -0.062859 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where Q is an orthogonal matrix. Solve the least squares problem $Ax = b$ with $b = [1\ 0\ 0\ 0]^T$ using the QR factorization of A . Show every step of the solution process.

9. (5p) Air density as a function of temperature is given in tabular form and the data is plotted as follows:



Suppose you desire the density at temperatures -80 and 80 degrees Centigrade, and these temperatures are not included in the table. Discuss what methods you can use, and whether they would be adequate or not.

10. (4p) True or false:
- If the $n \times n$ matrix H is a Householder reflector, and x is an arbitrary n -vector, then the last k components of the vector Hx are zero for some $k < n$.
 - The FFT algorithm can compute both the discrete Fourier transform and its inverse with the same efficiency.
11. (5p) A set of data was interpolated using the Fourier transform and the result was

$$\begin{aligned}
 P(t) = & 1.61 - 0.13 \cos 2\pi t - 0.50 \sin 2\pi t - 0.19 \cos 4\pi t - \\
 & 0.21 \sin 4\pi t - 0.20 \cos 6\pi t - 0.09 \sin 6\pi t - \\
 & 0.10 \cos 8\pi t
 \end{aligned}$$

Carry out the least squares fit of order 4.

12. (a) (2p) The Shannon information formula is

$$I = - \sum_{i=1}^k p_i \log_2 p_i$$

Calculate the average least number of bits needed to code the matrix

$$M = \begin{bmatrix} 0 & 8 & 8 & 3 \\ 0 & 4 & 4 & 3 \\ 3 & 4 & -1 & 4 \\ 4 & 3 & 8 & 3 \end{bmatrix}$$

- (b) (2p) Construct a Huffman tree for M .

(c) **(2p)** What is the average for this coding? What is the average if the standard binary system is used for the matrix entries?

13. **(5p)** A is a real matrix and when the *Matlab* command

`[X,L]=eig(A)`

is executed, the following result is obtained:

```
X = 0.3004      -0.73463      -0.73463      -0.57735
      0.042914   0.3025 + 0.25928i   0.3025 - 0.25928i   -0.57735
      0.55788    0.10803 - 0.06482i   0.10803 + 0.06482i   -0.57735
      0.77245    -0.51856 - 0.12964i   -0.51856 + 0.12964i   -1.6726e-015
```

```
L = 6      0      0      0
      0    -2.3592e-016 + 3i      0      0
      0      0    -2.3592e-016 - 3i      0
      0      0      0    -2.207e-015
```

- What algorithm was used by *Matlab*?
- What does matrix X contain?
- What does matrix L contain?
- Is it possible that matrix A is a symmetric matrix? Justify your answer.
- Is it possible to conclude that matrix A is invertible? Justify.

14. **(5p)** The eigenvalues of

$$S = \begin{bmatrix} 5 & 6 & 8 & 11 \\ 6 & 7 & 9 & 12 \\ 8 & 9 & 10 & 13 \\ 11 & 12 & 13 & 14 \end{bmatrix}$$

are -3.319, -0.5106, -0.02962 and 39.86. Find the singular values of S .